Factor investing: get your exposures right!

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ABSTRACT

This paper is devoted to the question of optimal portfolio construction for equity factor investing. The first part of the paper focusses on how to make sure that a given equity portfolio has the targeted factor exposures, even before imposing any constraints. We show that such portfolios can be derived from mean-variance optimization using stock expected returns as inputs provided these are built in a robust way from information about the factors. We propose a framework to build those robust stock expected returns and show that the targeted factor exposures are retained by the portfolios both before and after applying realistic constraints, e.g. long-only. Other more simplistic approaches fail. In the second part of the paper we illustrate the application of the framework to a practical case where the objectives are, first, to decide about the risk budget allocation to factors in some pragmatic way; and second, to construct a long-only constrained portfolio that retains the targeted exposures to four factors from well-known asset pricing equity models, namely High-minus-Low (HML), Robust-minus-Weak (RMW), Conservative-minus-Aggressive (CMA) and Momentum (MOM).

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1. INTRODUCTION

Factors are characteristics that explain the risk of stock and equity portfolio returns. According to financial theory from the 1950s consistent with the efficient market hypothesis, one factor, the market factor, should be all is needed. However, empirical evidence that the exposure to the market factor as measured by beta is not sufficient has been available since Haugen and Heines (1973) showed that US stocks sharing the common characteristic of exhibiting the lowest volatility delivered returns higher than expected from their level of beta. Basu (1977) showed that value stocks, those exhibiting the lowest price-to-earnings ratio, also delivered higher returns than expected from their level of beta. Jagadeesh and Titman (1993) showed that momentum stocks, those with stronger past performance at horizons of 12 months, also delivered average returns higher than predicted by their beta. More recently, Novy-Marx (2013) established that stocks that share the characteristic of being the most profitable, i.e. those with the highest gross profit, also delivered abnormally high returns in face of their beta.

Hou, Xue and Zhang (2017) listed 447 factors reported in financial literature believed to have explanatory power in the cross-section of stock returns. However, many of these factors overlap in terms of information content as they are just different ways of looking at the same characteristic. For that reason they can be grouped into just a handful of independent styles such as value, quality, momentum or low risk. Others, as pointed out by Hou et al. (2017), are simply too difficult to replicate or not statistically significant.

Indeed, at the other end of the scale, Fama and French (2015) proposed that a parsimonious asset pricing model for equities does not need more than five factors. Others, for example Blitz, Hanauer, Vidojevic and van Vliet (2018), proposed a number of factors not much larger.

Evidence that a myriad of factors generates premiums uncorrelated with the equity market factor abounds in academic literature, even if there is still no consensus when it comes to explaining the source of those premiums. Some explain them as compensation from exposures to risks; others explain them as resulting from behavioral preferences that create imbalances in the demand and offer of stocks sharing the common characteristics captured by a given factor.

Irrespective of the source of the factor premiums, strategies that promise outperformance over the market capitalization indices from tilts in favor of stocks exposed to factors like those typically grouped into value, quality, momentum and low risk styles have been growing in popularity. *Smart beta* strategies get there indirectly by starting with algorithms for portfolio construction which land in portfolios with exposures to factors that pay premiums. As shown by Leote de Carvalho et al. (2012), the risk and returns of the minimum variance and maximum diversification portfolios can be explained by tilts towards low-risk stocks, whereas the risk and returns to risk parity portfolios can be explained by the tilts towards the smaller capitalization stocks, low-risk stocks and, to some extent, value stocks. In turn, *factor investing* focusses on building portfolios with targeted factor exposures. The line dividing smart beta and factor investing is tenuous. But the portfolio construction algorithms and the factor exposures, acquired passively in smart beta and actively targeted in factor investing, tend to make the difference.

Different approaches have been put forward to building portfolios with multiple factor exposures. Haugen and Baker (1984), proposed the use of cross-sectional regressions applied to an arbitrary number of factors. The cross-sectional regression framework relies on past cross-sectional correlations of stock returns with the exposure of a stock to a factor in order to determine which factors are relevant and which are superfluous. In this framework, the last one-period cross-sectional stock returns are regressed against the stock factor exposures at the start of that period. The approach is repeated at each rebalancing, typically monthly or quarterly. The regression produces the optimal factor weights that should be used when generating forecasts of stock returns for the next period. The expected stock returns are then often used in mean-variance optimization to build the multi-factor portfolio. Unfortunately, as highlighted by Leote de Carvalho (2016), there are three problems. First, the framework is not workable when similar characteristics are used as factors, e.g. including factors from the same style, e.g. price-to-earnings and price-to-book as value factors. The problem stems from their strong correlation in the cross-section. Practitioners tend to dampen the cross-sectional factor correlations. However, this defeats the purpose. Ssecond, the weight of factors is volatile from one rebalancing to another unless averaged over a number of past periods, thus imposing factor weights to rely on momentum. Third, the stock expected returns are not robust and thus, when used in mean-variance optimization, lead to corner solutions, a problem solved in a unsatisfactory way by imposing constraints on stock weights.

Many practitioners prefer simpler approaches, e.g. multi-scoring approaches, just calculating an average multi-factor score for each stock based on the stock exposure to the factors chosen *a priori*. Factors are selected on the basis of empirical evidence of the premium they generated. The final portfolio is often just the equally-weighted selection of stocks with the highest average multi-factor scores, recalculated monthly or quarterly. However, the portfolios tend to exhibit systematic risk exposures not desired or controlled, e.g. over-reliance on smaller capitalization stocks, or a beta different from 1 and variable over time. Leote de Carvalho et al. (2017) and Amenc, Goltz and Lodh (2018) highlighted the importance of controlling for beta in multifactor portfolios. Moreover, with such a simplistic approach, it is not easy to manage portfolio constraints such as turnover or setting the beta to 1.

Finally, some prefer to consider the multi-factor score for each stock at each rebalancing as proxies of stock expected returns in mean-variance optimization. Unfortunately, as we shall see, this approach suffers from the same problems as using stock returns derived from multi-factor cross-sectional regressions: the optimal mean-variance portfolio exhibits a number of unwanted exposures to other risks.

One approach that became popular with practitioners is based on the idea that factor premiums can be captured efficiently with long-short portfolios that can be combined into a zero-sum long-short portfolio extension which, when sitting next to the benchmark index, generates tracking error and excess returns over that benchmark. This approach went by the name of 130/30 since often the active extension corresponded to a 30/30 long-short portfolio. As described by Lo and Patel (2008), such approaches capture well the excess returns from stocks with negative exposures to factors that can be difficult to underweight in long-only portfolios. However, despite their simplicity, interest from investors and practitioners faded considerably since the Global Financial Crisis because of counter-party risk and costs arising from the implementation of the extension portfolio using total return swaps.

One alternative by Leote de Carvalho et al. (2014) proposes that the stock implied returns derived by reverse optimization from the extension of 130/30 portfolios can be used in mean-variance optimization in order to efficiently build constrained multi-factor portfolios, e.g. long-only. They showed that: i) the implied stock returns are robust for mean-variance optimization; and ii) that their approach in effect minimizes the impact of portfolio constraints while retaining as much as possible the systematic factor risk exposures in the 130/30 portfolio from which the stock implied returns were derived.

In this paper, we focus on the question of optimal construction of multi-factor equity portfolios, in particular, how to make sure that the portfolio retains the desired factor exposures before and after applying constraints. As proposed by Leote de Carvalho et al. (2014), we split the problem into two. First, how to construct an optimal unconstrained multi-factor equity portfolio with the targeted factor exposures. Second, how to retain those factor exposures, even after applying constraints, e.g. long-only.

In Section 2, we consider three approaches for generating stock returns from factor returns. First, we consider a naïve approach, the simplest we could conceive, whereby stock expected returns are calculated as the sum-product of stock weights in each long-short factor portfolio by each respective expected factor return. This is a proxy of the multi-scoring approach. This approach runs into trouble because if we turn the problem around and try to calculate the factor returns from the derived stock returns using a similar naïve approach, we find a different result from the starting point, i.e. the problem is not reversible. In the other two examples, we impose the constraint that reversibility between stock and factor returns must hold. Since there is no unique way of imposing reversibility we consider two approaches: i) the simplest approach we could conceive and ii) the approach of Leote de Carvalho et al. (2014) that, indeed, satisfies the reversibility condition.

In Section 2, we also discuss the properties of unconstrained optimal mean-variance portfolios for each of the three approaches. We show that only the portfolios based on the approach of Leote de Carvalho et al. (2014) have no exposure to factors other than those targeted. In the other two cases, the portfolios show large exposures to factors orthogonal to those used to build the stock expected returns.

In Section 3, we show how to implement the approach of Leote de Carvalho et al. (2014). We discard the other two approaches in view of their lack of robustness. We first consider ways of deciding about the optimal factor weights. We show that the problem can be solved for by using simple matrix algebra as long as portfolios are unconstrained and we assume that the information ratio of all factors is the same. We considered i) taking into account factors correlations (Maximum Diversification) and ii) assuming that factors are uncorrelated (Equal Risk Budget). We also consider a third approach in which we seek the allocation to each factor such that the contribution to tracking error is the same from each (Equal Risk Contribution). This last approach can neither be resolved analytically nor considered mean-variance optimal under the simple hypothesis for the information ratio of factors. Nevertheless, it has its merits and may be considered by practitioners.

In Section 4, we illustrate the application of the framework with an example for the construction of active benchmarked long-only constrained equity multi-factor portfolios where we target positive exposures to four factors from well-known asset pricing equity models: High-minus-Low (HML), Robust-minus-Weak (RMW), Conservative-minus-Aggressive (CMA) and Momentum (MOM). We consider the three different risk budgeting allocations to factors discussed in Section 3 and we show the resulting portfolios. In particular, we demonstrate the robustness of the approach in allocating the desired factor risk budgets even when constraints are applied.

2. DERIVING STOCK RETURNS FROM FACTOR RETURNS

Suppose we have N stocks and K factors, the factors being zero sum linear combinations of stock weights, i.e. long-short portfolios, represented in a $\mathbf{P}(N * K)$ matrix with N rows and K lines. Let's set the objective of estimating expected stock returns from a given set of expected factor returns.

2.1. Naïve approach

The simplest approach to estimate stock expected returns $\mathbf{R}(1 * N)$ from the factor expected returns is to sum the product of the weights of each stock in the long-short portfolio for each factor by the factor expected returns $\mathbf{F}(1 * K)$, i.e. $\mathbf{R} = \mathbf{P} \cdot \mathbf{F}$. Let's introduce an.

Suppose we have a universe with N = 10 stocks and K = 3 factors at a given date and consider the matrix $\mathbf{P}(10 * 3)$ of zero sum long-short stocks weights for each factor portfolio as well as the vector of expected factor returns $\mathbf{F}(1 * 3)$. The vector of stock expected returns is given in Table 1.

This approach is similar to multi-scoring approaches used by practitioners, where the final stock return or score is a weighted average of the stock weight or score in each factor portfolio. Unfortunately this approach is not reversible. Indeed, if we re-calculate the factor returns from $\mathbf{F}_{new} = \mathbf{P}^T \cdot \mathbf{R}$, where T denotes transposed, the new factor expected returns differ from the starting factor expected returns. In Table 2, F_{new} / F is the ratio of these new to the initial factor expected returns.

R	Returns
Stock 1	9.03%
Stock 2	9.03%
Stock 3	3.74%
Stock 4	1.50%
Stock 5	3.80%
Stock 6	-3.74%
Stock 7	-1.50%
Stock 8	-3.80%
Stock 9	-9.03%
Stock 10	-9.03%

=

P	Factor 1	Factor 2	Factor 3
Stock 1	20%	20%	20%
Stock 2	20%	20%	20%
Stock 3	-20%	20%	20%
Stock 4	20%	-20%	20%
Stock 5	20%	20%	-20%
Stock 6	20%	-20%	-20%
Stock 7	-20%	20%	-20%
Stock 8	-20%	-20%	20%
Stock 9	-20%	-20%	-20%
Stock 10	-20%	-20%	-20%

X

F	Returns
Factor 1	13.24%
Factor 2	18.84%
Factor 3	13.08%

Table 1: Example of how stock returns can be naïvely estimated from factor portfolios and factor returns as a sum-product of the stock weight in each factor portfolio times the respective factor expected return, acting as a factor weight.

Moreover, the ratio between the new and the original factor returns is not the same for each factor, i.e. differences between initial and final factor expected returns are not just a matter of scaling: the expected return to Factor 2, the highest, is somewhat diluted when compared to the expected returns to the other two factors. Stock expected returns naïvely estimated in this way are not consistent with the factor returns.

$\mathbf{F}_{new}$	Returns
Factor 1	7.85%
Factor 2	9.64%
Factor 3	7.80%

F_{new}/F	Ratio
Factor 1	59.29%
Factor 2	51.18%
Factor 3	59.62%

Table 2: Expected factor returns derived from naïvely estimated stock expected returns.

2.2. Imposing consistency between stock and factor returns

Impose invariance of the expected factor returns relative to the transformations above requires an additional condition, namely that we seek stock expected returns so that the underlying factor expected returns remain unchanged, i.e. $\mathbf{F}_{new} = \mathbf{F}$. This condition is equivalent to imposing that *any residual return to a portfolio compared to a linear combination of the factors must have a zero expected return*, i.e. any portfolio orthogonal to the factors must have a zero expected return.

With this condition we have a well-posed problem with *N* equations and *N* unknown variables. The *N* equations are a function of the initial factor expected returns and the *N* - *K* null expected returns for the orthogonal basis to the factors.

The definition of orthogonality is arbitrary, a point we discuss later, in particular when looking at the consequences of different choices of the definition of orthogonality on portfolio optimization. For the moment, let's simply consider an arbitrary matrix $\mathbf{\Omega}(N * N)$. We want to estimate the stock expected returns $\mathbf{R}(1 * N)$ from a given vector of factor returns $\mathbf{F}(1 * K)$. \lLet $\mathbf{W}(1 * N)$ be any portfolio of stocks and $\mathbf{P}(N * K)$ the zero sum long-short weights for *N* stocks in each of the *K* factors, as before.

Let's project the portfolio $\mathbf{W}(1 * N)$ on the subspace defined by the factors, with the projection $\mathbf{A}(1 * N) = \mathbf{P} \cdot \mathbf{B}$, and the vector $\mathbf{B}(1 * K)$ representing the exposures of this portfolio **W** to the *K* factors.

We know that **A** is the unique portfolio linear combination of the *K* factors that minimizes the distance between **W** and **A,** i.e.:

$$\mathbf{B} = ArgMin[(\mathbf{W} - \mathbf{P} \cdot \mathbf{B})^T \mathbf{\Omega} (\mathbf{W} - \mathbf{P} \cdot \mathbf{B})] \quad (1)$$

Taking the derivative with respect to **W**:

$$\mathbf{P}^T \, \mathbf{\Omega} \, \mathbf{PB} = \mathbf{P}^T \, \mathbf{\Omega} \, \mathbf{W} \quad (2)$$

That is:

$$\mathbf{\Theta} \cdot \mathbf{B} = \mathbf{P}^T \, \mathbf{\Omega} \, \mathbf{W} \quad (3)$$

with $\mathbf{\Theta} = \mathbf{P}^T \, \mathbf{\Omega} \, \mathbf{P}$, the matrix of distances between factors.

As mentioned, we impose that the factor expected returns estimated from two possible ways match exactly for any portfolio **W**. This condition is translated into:

$$\mathbf{F}^T \cdot \mathbf{B} = \mathbf{R}^T \cdot \mathbf{W} \quad (4)$$

which in turn implies that the expected stock returns must be determined from the following function of the arbitrary matrix $\mathbf{\Omega}$:

$$\mathbf{R} = \mathbf{\Omega} \cdot \mathbf{P}\mathbf{\Theta}^{-1}\mathbf{F} \quad (5)$$

With $\boldsymbol{\beta}(N * K)$ the stock factors exposures defined as $\boldsymbol{\beta} = \mathbf{\Omega} \cdot \mathbf{P}\mathbf{\Theta}^{-1}$, equation (5) is just the standard formulation of a factor model for stock returns:

$$\mathbf{R} = \boldsymbol{\beta} \cdot \mathbf{F} \quad (6)$$

2.3. Calculation of the stock returns from factor returns

Let's examine the impact of choices for the arbitrary matrix $\mathbf{\Omega}$. The simplest choice is the identity matrix, i.e. $\mathbf{\Omega} = \mathbf{I}$. A second choice is to set $\mathbf{\Omega}$ to the variance-covariance matrix $\mathbf{\Sigma}$ of stock returns. As shown below, this choice is of particular interest if the stock expected returns derived from the formalism above are then used in a mean-variance optimizer for portfolio construction.

Let's run the example above through these choices of $\mathbf{\Omega}$. An example of variance-covariance matrix of the stock returns, $\mathbf{\Sigma} = \boldsymbol{\sigma}^T\boldsymbol{\rho}\boldsymbol{\sigma}$, defined from the vector of stock volatilities $\boldsymbol{\sigma}$ and the correlation matrix $\boldsymbol{\rho}$, is found in Table 3.

σ	Volatility
Stock 1	31.92%
Stock 2	21.73%
Stock 3	27.02%
Stock 4	21.33%
Stock 5	26.93%
Stock 6	20.13%
Stock 7	28.77%
Stock 8	25.31%
Stock 9	34.50%
Stock 10	20.97%

	Correlation									
ρ	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6	Stock 7	Stock 8	Stock 9	Stock 10
Stock 1	100%	80%	60%	60%	60%	40%	40%	40%	20%	20%
Stock 2	80%	100%	60%	60%	60%	40%	40%	40%	20%	20%
Stock 3	60%	60%	100%	40%	40%	20%	60%	60%	40%	40%
Stock 4	60%	60%	40%	100%	40%	60%	20%	60%	40%	40%
Stock 5	60%	60%	40%	40%	100%	60%	60%	20%	40%	40%
Stock 6	40%	40%	20%	60%	60%	100%	40%	40%	60%	60%
Stock 7	40%	40%	60%	20%	60%	40%	100%	40%	60%	60%
Stock 8	40%	40%	60%	60%	20%	40%	40%	100%	60%	60%
Stock 9	20%	20%	40%	40%	40%	60%	60%	60%	100%	80%
Stock 10	20%	20%	40%	40%	40%	60%	60%	60%	80%	100%

Table 3: Example of stock volatility of returns and pairwise correlation of stock returns. The correlations are based on the number of times the stock weights are in the same direction in each factor portfolio $\mathbf{P}$ as given above: 80% if three times, 60% if twice, 40% if once and 20% if none.

R	Naïve	Identity Matrix	Variance-Covariance Matrix
Stock 1	9.03%	16.13%	20.60%
Stock 2	9.03%	16.13%	13.41%
Stock 3	3.74%	7.64%	9.16%
Stock 4	1.50%	0.64%	1.05%
Stock 5	3.80%	7.84%	8.54%
Stock 6	-3.74%	-7.64%	-5.15%
Stock 7	-1.50%	-0.64%	0.74%
Stock 8	-3.80%	-7.84%	-6.17%
Stock 9	-9.03%	-16.13%	-20.07%
Stock 10	-9.03%	-16.13%	-11.41%

Table 4: Stock expected returns derived from the naïve approach and from two approaches where consistency with factor expected returns is imposed, one using the identity matrix and the other using the variance-covariance matrix of stock returns as measures of orthogonality.

The vector of stock expected returns, $\mathbf{R}$, obtained from the naïve approach and from the approaches where consistency is imposed with either $\mathbf{\Omega} = \mathbf{I}$ or $\mathbf{\Omega} = \mathbf{\Sigma}$ are given in table 4.

It is difficult to assess the advantage of imposing consistency via one or another choice of $\mathbf{\Omega}$. We simply note that when using $\mathbf{\Sigma}$, differences in stock expected returns are more pronounced, in particular for stocks 1 and 9, which have a higher volatility than stocks 2 and 10.

2.4. Optimal portfolios from stock returns

Let's compare the mean-variance optimal portfolios derived from the stock expected returns in Table 4 and $\mathbf{\Sigma}$ built from the volatility and correlations in Table 3. For a given level of risk aversion λ the mean-variance optimal portfolio $\mathbf{W}_{MV}$ is:

$$\mathbf{W}_{MV} = \frac{1}{\lambda} \mathbf{\Sigma}^{-1}\mathbf{R} \quad (7)$$

If λ is such that the volatility of the optimal portfolios is always 10%, we find the unconstrained optimal portfolios:

$\mathbf{W}_{MV}$	Naïve	Identity Matrix	Variance-Covariance Matrix
Stock 1	-4.66%	-4.08%	12.92%
Stock 2	36.63%	36.47%	12.92%
Stock 3	2.05%	2.87%	8.50%
Stock 4	6.80%	2.06%	-2.64%
Stock 5	6.27%	7.65%	7.07%
Stock 6	-17.31%	-18.99%	-8.50%
Stock 7	0.98%	2.05%	2.64%
Stock 8	-7.46%	-8.14%	-7.07%
Stock 9	9.95%	10.14%	-12.92%
Stock 10	-40.99%	-39.34%	-12.92%
Sum	-7.74%	-9.31%	0.00%

Table 5: Mean variance optimal portfolios derived from the different stock expected returns in Table 4 for a level of volatility at 10% in each case.

The first two portfolios are similar and suffer from similar undesirable properties. First, the sum of weights is not 0. Second, stock 1 is sold short and stock 9 is bought while the expected return for stock 1 is positive and for stock 9 is negative.

The third case is special. The fact that we chose $\mathbf{\Omega} = \mathbf{\Sigma}$ implies that the matrix of distances between factors, $\mathbf{\Theta}$, is now the variance-covariance matrix of the factor portfolios themselves, $\mathbf{\Theta} = \mathbf{P}^T \, \mathbf{\Sigma} \, \mathbf{P}$. For this reason, the third portfolio is an exact linear combination of the three factor portfolios. Indeed, since the optimal mean-variance portfolio is proportional to $\mathbf{\Sigma}^{-1}\mathbf{R}$ in terms of stocks, then this optimal portfolio is also proportional to $\mathbf{P\Theta}^{-1}\mathbf{F}$ in terms of factors, as implied by equation (5). The proportion of each factor portfolio is determined by a mean-variance optimization of factors as $\lambda^{-1}\mathbf{\Theta}^{-1}\mathbf{F}$. Both the mean-variance optimal portfolio of factors and the mean variance optimal portfolio of stocks exhibit the same stock allocation. Because the factor portfolios are zero sum long-short portfolios, the third portfolio is itself a zero sum long-short portfolio.

2.5. Factor contribution towards optimal portfolio variance

We now look the exposures of the optimal portfolios in Table 5 to the three factor portfolios in Table 1. To do this, we need estimate the vector $\mathbf{B}$ with the portfolios' exposures to those factors. This can be derived from equation (3) as $= \mathbf{\Theta}^{-1}\mathbf{P}^T \, \mathbf{\Omega} \, \mathbf{W}$.

From the factor exposures $\mathbf{B}$ and the stock volatilities and correlations in Table 3, we can decompose the variance of each optimal portfolio into i) the contribution from each of the three factor portfolios, ii) a contribution from the correlation of those factors, and ii) the difference, which is essentially an exposure to other factors orthogonal to the three factors considered here.

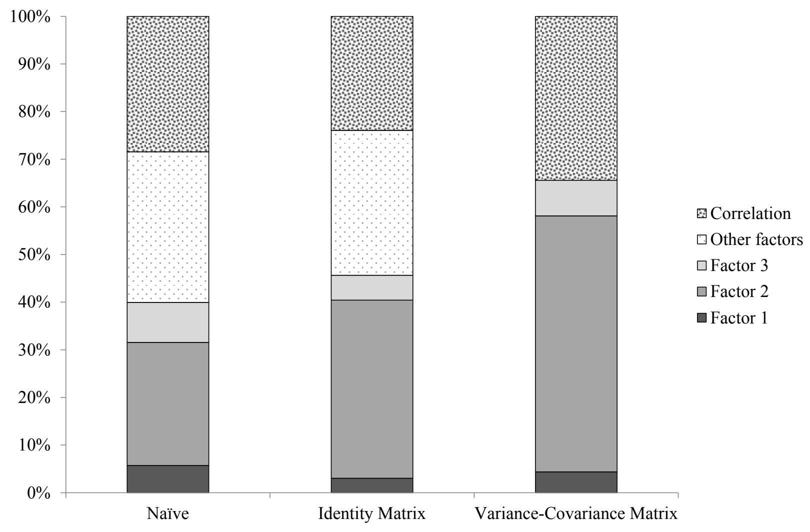

Figure 1: Variance decomposition for the optimal portfolios in Table 5

The first two approaches result in portfolios with a large component of their risk derived from exposures to other factors, which have zero expected returns because they are orthogonal to the three factors considered. That is a non-desirable property of these approaches. The third approach, in which consistency is imposed by using $\mathbf{\Sigma}$ as the arbitrary matrix in equation (5), has no exposure to orthogonal risk factors and thus all the risk budget in the portfolio is used sensibly, i.e. either for exposure to factors with a positive return or for diversification.

2.6. Adding a correlated factor

To test for robustness, we now add a factor 4, highly correlated with factor 1. The difference between factor 4 and factor 1 is represented by $\Delta_{F4\text{-}F1}$, a long-short portfolio invested in stock 1 and short selling stock 2 by just 1%.

P	Factor 1	Factor 2	Factor 3	Factor 4	$\Delta_{F4\text{-}F1}$
Stock 1	20%	20%	20%	21%	1%
Stock 2	20%	20%	20%	19%	-1%
Stock 3	-20%	20%	20%	20%	0%
Stock 4	20%	-20%	20%	20%	0%
Stock 5	20%	20%	-20%	-20%	0%
Stock 6	20%	-20%	-20%	-20%	0%
Stock 7	-20%	20%	-20%	-20%	0%
Stock 8	-20%	-20%	20%	20%	0%
Stock 9	-20%	-20%	-20%	-20%	0%
Stock 10	-20%	-20%	-20%	-20%	0%

Table 6: Highly correlated factor added to the set of factor portfolios in Table 1. Factor 4 differs from factor 1 in terms of stock weights in the portfolio as represented by $\Delta_{F4\text{-}F1}$.

In addition to the factor expected returns in Table 1, we consider different assumptions for the return to factor 4. First, the return to the factor 4 equals the return to factor 1. Second, stock 1 outperforms stock 2 by 10%, i.e. the return to factor 4 exceeds the return to factor 1 by 0.1%. Third, stock 1 outperforms stock 2 by 100%, i.e. the return to factor 4 exceeds the return to factor 1 by 1.0%.

2.6.1. Impact of adding correlated factor on stock expected returns

We shall now investigate how this correlated factor impacts the stock expected returns, the optimal portfolios and their respective variance decomposition. We start by looking at the impact on the stock expected returns.

2.6.1.1. Stock expected returns with naïve approach

In table 7 we show that for the naïve approach the magnitude of the outperformance of factor 4 over factor 1 has almost no impact on the stock expected returns. The weight of factor 1 is simply multiplied by two in the expectations. However, if instead of factor 4, we had added a factor Δ_{F4-F1} directly, then the stock expected returns would have been modified.

R	3 factors	4 factors		
		R_{F4}-R_{F1}=0	R_{F4}-R_{F1}=0.1%	R_{F4}-R_{F1}=1.0%
Stock 1	9.03%	11.81%	11.83%	12.02%
Stock 2	9.03%	11.55%	11.57%	11.74%
Stock 3	3.74%	1.09%	1.07%	0.89%
Stock 4	1.50%	4.14%	4.16%	4.34%
Stock 5	3.80%	6.45%	6.47%	6.65%
Stock 6	-3.74%	-1.09%	-1.07%	-0.89%
Stock 7	-1.50%	-4.14%	-4.16%	-4.34%
Stock 8	-3.80%	-6.45%	-6.47%	-6.65%
Stock 9	-9.03%	-11.68%	-11.70%	-11.88%
Stock 10	-9.03%	-11.68%	-11.70%	-11.88%

Table 7: Stock expected returns derived from applying the naïve approach to the factor returns given in Table 1 and the factor weights defined in Table 6.

2.6.1.2 Stock expected returns with identity matrix

The results are more satisfactory when the identity matrix is used because the new factor does not change the return expectations for stocks 3 to 10, but rather it adjusts the expected returns to stocks 1 and 2 accordingly so that their return difference is consistent with that implied by the expected return for Δ_{F4-F1}.

R	3 factors	4 factors		
		R_{F4}-R_{F1}=0	R_{F4}-R_{F1}=0.1%	R_{F4}-R_{F1}=1.0%
Stock 1	16.13%	16.13%	21.13%	66.13%
Stock 2	16.13%	16.13%	11.13%	-33.87%
Stock 3	7.64%	7.64%	7.64%	7.64%
Stock 4	0.64%	0.64%	0.64%	0.64%
Stock 5	7.84%	7.84%	7.84%	7.84%
Stock 6	-7.64%	-7.64%	-7.64%	-7.64%
Stock 7	-0.64%	-0.64%	-0.64%	-0.64%
Stock 8	-7.84%	-7.84%	-7.84%	-7.84%
Stock 9	-16.13%	-16.13%	-16.13%	-16.13%
Stock 10	-16.13%	-16.13%	-16.13%	-16.13%

Table 8: Stock expected returns derived from imposing consistency with the factor returns given in Table 1 and the factor weights defined in Table 6 when using the identity matrix as the orthogonality measure.

2.6.1.3 Stock expected returns with variance-covariance matrix

The expected returns are now more difficult to interpret. The imposed wider outperformance of stock 1 over stock 2 is easy to spot in the third and fourth cases. In the second case, it is also easy to see the consistency from imposing that stocks 1 and 2 have the same expected return if the difference in the returns of factor 4 and 1 is zero. In each case, the returns to other stocks are adjusted so as not to deviate too strongly from the expected return for stock 1.

R	3 factors	4 factors		
		R_{F4}-R_{F1}=0	R_{F4}-R_{F1}=0.1%	R_{F4}-R_{F1}=1.0%
Stock 1	20.60%	13.46%	23.39%	112.74%
Stock 2	13.41%	13.46%	13.39%	12.74%
Stock 3	9.16%	6.87%	10.06%	38.79%
Stock 4	1.05%	-1.44%	2.03%	33.29%
Stock 5	8.54%	5.90%	9.57%	42.69%
Stock 6	-5.15%	-7.66%	-4.16%	27.31%
Stock 7	0.74%	-1.98%	1.80%	35.80%
Stock 8	-6.17%	-9.04%	-5.05%	30.80%
Stock 9	-20.07%	-24.41%	-18.37%	35.95%
Stock 10	-11.41%	-13.94%	-10.42%	21.23%

Table 9: Stock expected returns derived from imposing consistency with the factor returns given in Table 1 and the factor weights defined in Table 6 when using **Σ** as the orthogonality measure.

2.6.2. Impact of adding a correlated factor to optimal portfolios

Let's look at the mean-variance optimal portfolios generated from the stock expected returns when the correlated factor is included.

2.6.2.1. Optimal portfolios with the naïve approach

The addition of a fourth correlated factor has a small impact on the optimal portfolios. Changing the expected outperformance of factor 4 over factor 1 has little impact, reflecting the lack of consistency between factor and stock expected returns.

$\mathbf{W}_{MV}$	3 factors	4 factors		
		R_{F4}-R_{F1}=0	R_{F4}-R_{F1}=0.1%	R_{F4}-R_{F1}=1.0%
Stock 1	-4.66%	-4.73%	-4.72%	-4.72%
Stock 2	36.63%	33.90%	33.87%	33.64%
Stock 3	2.05%	-1.65%	-1.67%	-1.86%
Stock 4	6.80%	12.46%	12.49%	12.74%
Stock 5	6.27%	6.36%	6.36%	6.34%
Stock 6	-17.31%	-2.97%	-2.89%	-2.14%
Stock 7	0.98%	-0.63%	-0.64%	-0.72%
Stock 8	-7.46%	-11.21%	-11.23%	-11.39%
Stock 9	9.95%	8.57%	8.56%	8.46%
Stock 10	-40.99%	-43.22%	-43.22%	-43.21%
Sum	-7.74%	-3.12%	-3.09%	-2.86%

Table 10: Mean-variance portfolio obtained from the stock expected returns in Table 7 when using the naïve approach.

Indeed, the larger difference in expected returns between factor 4 and factor 1 barely changes the model portfolio, even when this difference increases to 1%. Finally, as seen in Table 7, comparable expected returns for both stocks 1 and 2 lead to a larger and positive weight in stock 2, the least volatile of the two.

2.6.2.2. Optimal portfolios with the identity matrix

As shown in Table 11 below, nothing changes when factor 4 and factor 1 have the same expected returns. When the returns for both stocks 1 and 2 are the same, the portfolio puts more weight in stock 2, the least volatile of the two. But as the difference in returns between the two factors increases, as shown in Table 8, the increasingly large difference between the expected returns for stocks 1 and 2 translates into a greater difference in portfolio weights now favorable to stock 1.

$\mathbf{W}_{MV}$	3 factors	4 factors		
		R_{F4}-R_{F1}=0	R_{F4}-R_{F1}=0.1%	R_{F4}-R_{F1}=1.0%
Stock 1	-4.08%	-4.08%	18.37%	46.49%
Stock 2	36.47%	36.47%	-1.08%	-74.09%
Stock 3	2.87%	2.87%	4.76%	4.30%
Stock 4	2.06%	2.06%	4.38%	5.11%
Stock 5	7.65%	7.65%	9.77%	5.36%
Stock 6	-18.99%	-18.99%	-19.36%	-3.11%
Stock 7	2.05%	2.05%	2.49%	1.18%
Stock 8	-8.14%	-8.14%	-8.11%	-0.95%
Stock 9	10.14%	10.14%	9.82%	0.56%
Stock 10	-39.34%	-39.34%	-42.44%	-11.34%
Sum	-9.31%	-9.31%	-21.41%	-26.48%

Table 11: Mean-variance portfolio obtained from the stock expected returns in Table 8 when using the identity matrix as the orthogonality measure.

2.6.2.3. Optimal portfolios with the variance-covariance matrix

As discussed in section 2.4, in this case the optimal mean-variance portfolio, $\mathbf{W}_{MV}$, is proportional to a weighted average of all factor portfolios, $\mathbf{P}$, as implied by equation (5), with the weights of each factor portfolios given by $\mathbf{\Theta}^{-1}\mathbf{F}$, where the matrix $\mathbf{\Theta}$ is now the variance-covariance of the factor portfolios. This is a mean-variance optimization in the space of factors returns and factors variance-covariance determining the optimal weight of each factor portfolio in the final optimal stock portfolio. It is thus not surprising that with three factors, stock 1 and 2 have the same weight since they have the same weights in factors 1, 2 and 3. When factor 4 is introduced and factor 1 and 4 have the same expected return, the factor allocation is strongly tilted in favor of the less volatile factor 1 and turns negative against the almost identical but more volatile factor 4. This explains the large overweight of stock 2 against stock 1. But in the case where factor 4 has a higher expected return that of factor 1, the optimal factor allocation will increasingly arbitrage factor 4 in favor of factor 1 as the difference in the factor returns increases. This explains the increasing difference in the weight of stock 1 against the weight of stock 2.

$\mathbf{W}_{MV}$	3 factors	4 factors		
		R_{F4}-R_{F1}=0	R_{F4}-R_{F1}=0.1%	R_{F4}-R_{F1}=1.0%
Stock 1	12.92%	-4.96%	19.84%	51.30%
Stock 2	12.92%	32.60%	4.36%	-55.52%
Stock 3	8.50%	9.10%	7.95%	-1.44%
Stock 4	-2.64%	-2.52%	-2.60%	-0.44%
Stock 5	7.07%	7.23%	6.75%	-0.23%
Stock 6	-8.50%	-9.10%	-7.95%	1.44%
Stock 7	2.64%	2.52%	2.60%	0.44%
Stock 8	-7.07%	-7.23%	-6.75%	0.23%
Stock 9	-12.92%	-13.82%	-12.10%	2.11%
Stock 10	-12.92%	-13.82%	-12.10%	2.11%
Sum	0.00%	0.00%	0.00%	0.00%

Table 12: Mean-variance portfolio obtained from the stock expected returns in Table 8 when using the variance-covariance matrix as the orthogonality measure.

2.6.3. Impact of adding a correlated factor in portfolio variance decomposition

Finally, we look at the variance decomposition for the portfolios in Tables 10 through 12.

2.6.3.1. Variance decomposition with the naïve approach

In line with the portfolios in Table 10, the factor exposures of the portfolios constructed from four factors are comparable. Also, the allocation to factor 1 increases significantly when factor 4 is added. As a consequence, the last three portfolios in Table 10 are similar – there is little difference in the risk exposures of those portfolios. A final observation is the fact that all portfolios in Figure 2 exhibit exposure to other orthogonal factors that by definition have zero expected returns.

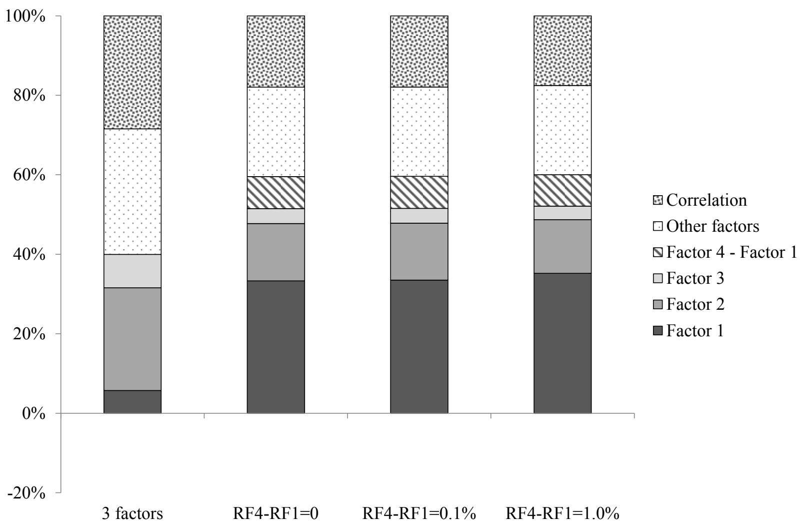

Figure 2: Variance decomposition of the mean-variance portfolio obtained from the stock expected returns in Table 8 when using the naïve approach.

2.6.3.2. Variance decomposition with the identity matrix

When the identity matrix is used, the factor exposures in the portfolios change significantly. In the last case, the risk exposure to the difference between factor 4 and factor 1 dominates the risk of the optimal portfolio, perhaps not surprisingly. The risk-adjusted return from investing in factor 4 financed from shorting factor 1 is increasingly large as the difference in returns between each increases because the expected volatility from that long-short remains small thanks to the large correlation of their returns. The optimal portfolio is thus increasingly positioned to capture the increasingly large risk-adjusted returns generated from shorting factor 1 to invest in factor 4. Unfortunately, all portfolios still have significant risk exposures to other factors for which expected returns are zero.

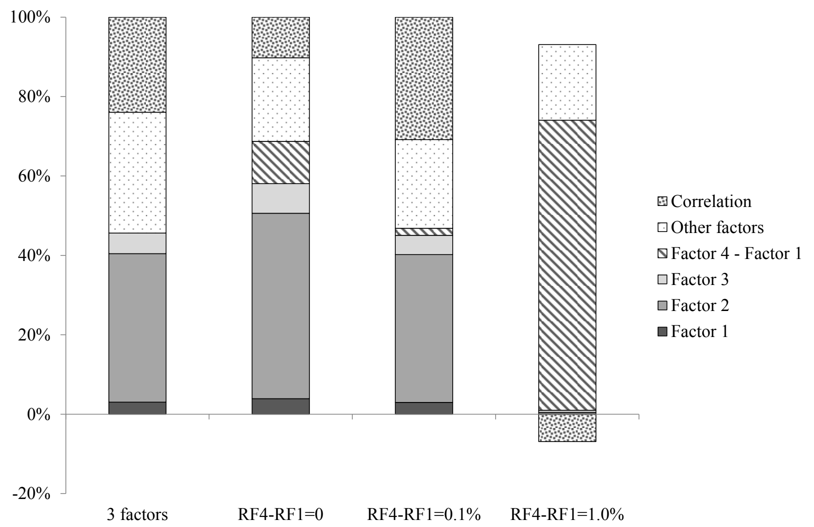

Figure 3: Variance decomposition of the mean-variance portfolio obtained from the stock expected returns in Table 8 when using the identity matrix as the orthogonality measure.

2.6.3.3. Variance decomposition with the variance-covariance matrix

Here, the risk exposures of each portfolio are aligned with those found in Figure 3 when the identity matrix was used instead. The key difference is that no exposure to other factors can be found here, which is an important advantage of using $\mathbf{\Sigma}$ instead of $\mathbf{I}$ as the orthogonality measure. The portfolios are fully exposed to factors with positive expected returns and have zero exposure to factors with zero expected returns. As before, the optimal portfolio is heavily exposed to the difference between factor 4 and factor 1, profiting from the increasing large risk-adjusted returns derived from the arbitrage from these two highly correlated factor portfolios.

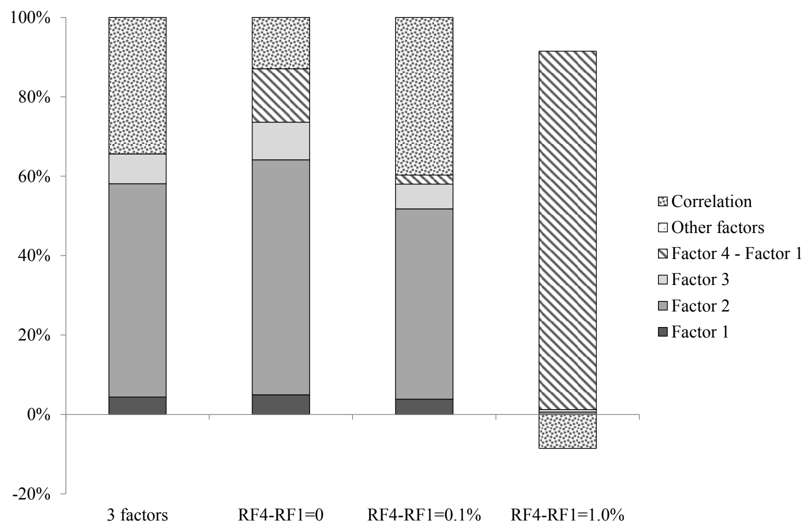

Figure 4: Variance decomposition of the mean-variance portfolio obtained from the stock expected returns in Table 8 when using the variance-covariance matrix as the orthogonality measure.

2.7. Implications

Section 2 shows that portfolios derived from the naïve approach i) have exposures to factors orthogonal to those targeted and ii) rely on stock returns that are inconsistent with the returns to the targeted factors. When the $\mathbf{I}$ is used as measure of orthogonality to impose consistency between factor and stock returns, we still find a significant exposure to factors orthogonal to those considered. Moreover, such factors must have zero expected return, which means that the risk allocation to such factors is not expected to pay. This attempt to solve for the inconsistency between factor and stock returns does not deliver portfolios that make sense. It is only when $\boldsymbol{\Sigma}$ is used as measure of orthogonality that the portfolios are fully exposed only to factors with positive expected returns. For this reason, we shall only consider this case in the remainder of this paper. Finally, we also showed that in this case, the exposure to factors in the optimal stock portfolio is determined by a mean-variance allocation to the factors based on their expected returns, volatility and correlations.

3. FACTOR WEIGHTS AND FACTOR EXPOSURES

3.1. The general case

We shall now focus only on $\boldsymbol{\Omega} = \boldsymbol{\Sigma}$. As discussed in Section 2.6.2.3, the weights of each factor portfolio are determined from solving a mean-variance optimization problem based on the factor expected returns and the variance-covariance matrix $\boldsymbol{\Theta}$ of the factor returns. The mean-variance optimal stock allocation derived from the stock expected returns or from the factor expected returns is the same since from (5) we can write:

$$\mathbf{W}_{MV} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\mathbf{R} = \frac{1}{\lambda}\mathbf{P}(\boldsymbol{\Theta}^{-1}\mathbf{F}) \qquad (8)$$

Here $\boldsymbol{\Theta}^{-1}\mathbf{F}/\lambda$ is the optimal mean-variance weight allocation to each factor, λ is the same parameter used in (7) measuring the overall risk aversion. This can be scaled so that the ex-ante volatility is set a target level. The total risk budget allocation is inversely proportional to the overall risk aversion.

Since factor portfolios are long-short portfolios, the optimal portfolio $\mathbf{W}_{MV}$ is also a long-short portfolio, as discussed in section 2.4. If we use the approach to construct fully invested benchmark portfolios, then $\mathbf{W}_{MV}$ in equation (8) represents the optimal stock active weight allocation and adds to zero. The volatility of this portfolio then becomes the tracking error against the benchmark.

The risk budget allocated to a factor is by definition the factor weight times the factor volatility. It is useful to think in terms of risk budget allocation to factors instead of weight allocation to factors because, whereas the solution to the mean-variance factor allocation problem in terms of factor weights is a function of the leverage applied to each factor portfolio, the solution in terms of factor risk budget is invariant. That is because the volatility of the factor portfolios scales linearly with the leverage of the underlying factor long-short portfolios.

If $\mathbf{RB}_{MV}$ is the vector of mean–variance optimal risk budget allocated to each factor, $\mathbf{IR}$ the vector of expected information ratio for each factor and $\boldsymbol{\rho}_f$ the correlation matrix of the factor returns, then:

$$\mathbf{RB}_{MV} = \frac{1}{\lambda}\boldsymbol{\rho}_f^{-1}\,\mathbf{IR} \qquad (9)$$

The mapping of optimal factor risk budgets into optimal stock weights is:

$$\mathbf{W}_{MV} = \mathbf{P}\,\mathbf{RB}_{MV} \qquad (10)$$

Equation (9) is the mean–variance unconstrained optimization problem of allocating to factors re-written in terms of risk budget allocation to factors, from factor correlations and factor information ratios, instead of the more conventional formulation in terms of factor weights, from factor variance-covariance and factor returns. Equation (9) is invariant to changes in the volatility of factor returns, i.e. invariant to the choice of leverage of the underlying factor portfolios.

Moreover, if the volatility of all factors is the same then the expected return of each stock is multivariate beta times expected factor return. To show this we can consider first the reverse optimization problem derived from (8) and (10):

$$\mathbf{R} = \lambda\,\boldsymbol{\Sigma}\,\mathbf{W}_{MV} = \lambda\,\boldsymbol{\Sigma}\,\mathbf{P}\,\mathbf{RB}_{MV} \qquad (11)$$

which, using (9), is:

$$\mathbf{R} = \mathbf{\Sigma}\,\mathbf{P}\,\boldsymbol{\rho}_f^{-1}\,\mathbf{IR} \tag{12}$$

As shown in the appendix, when factors have the same volatility then the multivariate exposure of each stock to the factors is:

$$\boldsymbol{\beta}_m \propto \mathbf{\Sigma}\,\mathbf{P}\,\boldsymbol{\rho}_f^{-1} \tag{13}$$

From (11) and (12), taking into account that factors have the same volatility, the stock expected return is beta multivariate times expected factor return:

$$\mathbf{R} \propto \boldsymbol{\beta}_m\,\mathbf{F} \tag{14}$$

3.2. Maximum Diversification, Equal Risk Budget and Equal Risk Contribution

We focused on building a robust optimal portfolio exposed to factors we expect to pay positive returns. In order to do so, in particular when constraints apply to the portfolio, we used stock expected returns derived from the factor expected returns as optimization inputs to find the optimal stock portfolio with the desired factor exposures. The framework proposed, while robust, is nevertheless sensitive to estimation errors in the factor returns. The higher the correlation of factors returns the more likely estimation errors may cause trouble as shown in Section 2.6.3.3. Therefore, we now introduce some sensible solutions avoiding the need for explicit forecasts of factor returns.

In the first example, we just consider that all factor information ratios are positive and equal. In this case, the allocation to factors when the mean-variance approach is applied follows what is known as Maximum Diversification, or MD, first introduced by Choueifaty and Coignard (2008).

The risk budget allocation in MD can be derived from (9), assuming that all the factor information ratios are equal. When the volatility of all factors is equal we find:

$$\mathbf{RB}_{MD} \propto \boldsymbol{\rho}_f^{-1}\,\mathbf{1} \tag{15}$$

$\mathbf{1}$ is the unit vector. Indeed, the optimal risk budget allocation minimizes correlation, allocating higher risk budgets to factors with the lowest correlations and lower risk budgets to factors more correlated with others. In turn, if all factor information ratios are equal, then, when the volatility of all factors is equal, the factor returns are also equal and from (14) we find the stock expected returns:

$$\mathbf{R}_{MD} \propto \boldsymbol{\beta}_m\,\mathbf{1} \tag{16}$$

For MD, the risk budget allocated to a factor can be negative even if its expected return is positive. By construction, the MD approach, when unconstrained, generates the same active univariate exposures for all factors, a result of the underlying assumption that the information ratio of all factors is equal.

The equal-risk budget, or ERB, simplification assumes that the risk budget allocation to all factors is the same:

$$\mathbf{RB}_{ERB} \propto \mathbf{1} \tag{17}$$

ERB is mean-variance efficient when all factor correlations are zero and information ratios equal. From (11) and (17), the stock expected returns are:

$$\mathbf{R}_{ERB} \propto \mathbf{\Sigma}\,\mathbf{P}\;\mathbf{1} \tag{18}$$

For ERB, the risk budget allocated to each factor is always positive but the factor exposures can be negative. Because all factor portfolios have the same volatility, the weight allocated to each factor is the same.

A third approach that falls somewhere in between the two above is to allocate a risk budget to each factor so that their contribution to risk is the same. We call this equal risk contribution, or ERC, an idea first proposed by Maillard, Roncalli and Teïletche (2010). The solution is no longer mean-variance efficient for any simple choice of factor information ratios and correlations. But this, nevertheless, constitutes a practical alternative approach to allocate to factors. Both the factor risk budget and the factor exposures are now positive. Since the factor volatilities are equal, the targeted ERC factor risk budget allocation can be numerically solved using:

$$\mathbf{RB}_{ERC} \propto \operatorname{argmin}\left[\sum_i \sum_j \left(RB_i\left(\boldsymbol{\rho}_f\,\mathbf{RB}\right)_i - RB_j\left(\boldsymbol{\rho}_f\,\mathbf{RB}\right)_j\right)^{\mathbf{2}}\right] \tag{19}$$

with the constraint that all risk budgets are positive. Once the factor risk budgets are found numerically, the optimal unconstrained stock allocation can be found from (10). The underlying stock expected returns can be calculated from the stock portfolio in (8) using reverse optimization:

$$\mathbf{R}_{ERC} \propto \mathbf{\Sigma}\,\mathbf{W}_{MV} \propto \mathbf{\Sigma}\,\mathbf{P}\,\mathbf{RB}_{\boldsymbol{ERC}} \tag{20}$$

By construction, and since the factor volatilities are equal, the weight of each factor is such that the product of the weight of each factor by the respective portfolio univariate exposure to the factor is equal for all factors.

In Table 13 we give the most important properties of the different choices to simplify the problem of allocating to factors.

	MD	ERB	ERC
Factor weight	positive or negative	positive and equal for all factors	(factor weight x factor exposure) equal for all factors
Univariate factor exposure	positive and equal for all factors	positive or negative	positive

Table 13: Properties of the MD, ERB and ERC choices of allocation to factors.

4. APPLICATION TO CONSTRAINED MULTI-FACTOR PORTFOLIOS

Let's consider an example. The objective is to construct a fully invested equity multi-factor portfolio where the tracking error relative to the benchmark index derives from the factor active exposures. For this reason, we shall consider $\mathbf{W}_{MV}$ as the zero-sum portfolio with optimal active stock weights and the volatility of which is the tracking error risk of the benchmarked portfolio. If we add the benchmark index to $\mathbf{W}_{MV}$ we find the optimal stock weights in the fully invested benchmarked portfolio, i.e. with stock weights totaling 100%.

We consider the unconstrained case, where the portfolio may include short positions. This is the case if the size of the short of a given stock in $\mathbf{W}_{MV}$ exceeds the weight of the stock in the benchmark portfolio. For this reason, we also include the case of the long-only constrained fully invested portfolio where no short positions are allowed. This portfolio can be obtained numerically by solving equation (8) under long-only constraints from the vector of stock expected returns $\mathbf{R}$ that can be obtained from (20) in both the general case and the ERC case, and simplifies into (16) and (18) in the MD and ERB cases, respectively.

In the example, we use the the Stoxx 50 index on July 21st 2017 and construct factor portfolios using four known factors: High-minus-Low (HML), Robust-minus-Weak (RMW) and Conservative-minus-Aggressive (CMA), from the factor model proposed by Fama and French (2015), and Momentum (MOM), a factor proposed by Carhart (1997). The first three use price-to-book, gross-profit and asset growth as indicators. The last uses the historical return of stocks over 11 months calculated one month before the date of the portfolio construction. We shall consider the three cases of factor active allocation discussed above: maximum diversification, equal risk budgeting and equal risk contribution.

			Factors				Long-short factor portfolios				Stock active weights					
											MD		ERB		ERC	
Company name	GICS Sector	Market weight	Price to Book	Gross Margin	Asset Growth	12M-1M	HML	RMW	CMA	MOM	uncons-trained	long only	uncons-trained	long only	uncons-trained	long only
Adidas AG	CD	1.4%	4.7	47%	14%	24%	-2.1%	3.2%	-2.6%	3.1%	1.9%	2.5%	0.8%	-1.2%	1.1%	0.2%
LVMH Moet Hennessy Louis Vuitton SE	CD	2.5%	3.5	65%	3%	66%	0.0%	3.2%	2.6%	3.1%	3.7%	5.0%	4.0%	5.0%	4.0%	5.0%
Vivendi SA	CD	0.9%	1.3	37%	-7%	23%	0.0%	0.0%	2.6%	0.0%	0.1%	-0.9%	1.1%	0.1%	0.9%	-0.9%
Volkswagen AG Pref	CD	1.1%	0.8	19%	7%	12%	2.1%	-3.2%	0.0%	-3.1%	-2.1%	-1.1%	-1.9%	-1.1%	-2.0%	-1.1%
Daimler AG	CD	2.7%	1.3	21%	12%	15%	0.0%	-3.2%	0.0%	0.0%	-2.0%	-2.7%	-1.4%	-2.7%	-1.7%	-2.7%
Bayerische Motoren Werke AG	CD	1.1%	1.2	24%	9%	15%	2.1%	0.0%	0.0%	0.0%	1.5%	1.6%	0.9%	0.0%	1.0%	0.6%
Industria de Diseno Textil, S.A.	CD	1.6%	7.5	28%	13%	15%	-2.1%	0.0%	-2.6%	-3.1%	-3.2%	-1.6%	-3.4%	-1.6%	-3.3%	-1.6%
L'Oreal SA	CS	1.9%	4.0	72%	6%	10%	-2.1%	3.2%	2.6%	3.1%	2.2%	5.0%	3.0%	5.0%	2.9%	5.0%
Anheuser-Busch InBev SA/NV	CS	3.1%	2.9	58%	98%	-10%	0.0%	3.2%	-2.6%	-3.1%	0.3%	-3.1%	-1.1%	-3.1%	-0.6%	-3.1%
Royal Ahold Delhaize N.V.	CS	0.9%	1.6	27%	133%	-19%	2.1%	-3.2%	-2.6%	-3.1%	-2.2%	-0.9%	-3.0%	-0.9%	-2.9%	-0.9%
Danone SA	CS	1.7%	2.8	51%	36%	6%	2.1%	0.0%	0.0%	0.0%	1.5%	-1.7%	0.9%	-1.7%	1.0%	-1.7%
Unilever NV Cert. of shs	CS	3.3%	6.8	43%	8%	23%	-2.1%	-3.2%	2.6%	3.1%	-1.8%	-3.3%	0.2%	-3.3%	-0.4%	-3.3%
Total SA	EN	4.5%	1.3	24%	6%	8%	-2.1%	-3.2%	-2.6%	3.1%	-2.1%	-4.5%	-2.1%	-4.5%	-2.2%	-4.5%
Eni S.p.A.	EN	1.4%	1.1	36%	-11%	1%	2.1%	3.2%	2.6%	-3.1%	2.1%	4.4%	2.1%	5.0%	2.2%	5.0%
Intesa Sanpaolo S.p.A.	FN	1.8%	0.9	NA	7%	43%	0.0%	0.0%	-2.6%	0.0%	-0.1%	-1.8%	-1.1%	-1.8%	-0.9%	-1.8%
Allianz SE	FN	3.5%	1.1	NA	4%	44%	-2.1%	0.0%	-2.6%	0.0%	-1.6%	-3.5%	-2.0%	-3.5%	-1.9%	-3.5%
Munich Reinsurance Company	FN	1.2%	0.9	NA	0%	27%	0.0%	0.0%	0.0%	-3.1%	-1.6%	-1.2%	-1.4%	-1.2%	-1.4%	-1.2%
Banco Bilbao Vizcaya Argentaria, S.A.	FN	2.1%	0.9	NA	-3%	43%	0.0%	0.0%	2.6%	0.0%	0.1%	-2.1%	1.1%	-2.1%	0.9%	-2.1%
Banco Santander S.A.	FN	3.9%	0.8	NA	0%	55%	2.1%	0.0%	2.6%	3.1%	3.2%	5.0%	3.4%	5.0%	3.3%	5.0%
Deutsche Bank AG	FN	1.2%	0.4	NA	-2%	29%	2.1%	0.0%	2.6%	-3.1%	0.1%	-1.2%	0.7%	-1.2%	0.5%	-1.2%
Societe Generale S.A. Class A	FN	1.6%	0.6	NA	4%	60%	2.1%	0.0%	0.0%	3.1%	3.1%	4.2%	2.3%	4.4%	2.4%	5.0%
AXA SA	FN	2.2%	1.0	NA	1%	40%	-2.1%	0.0%	0.0%	-3.1%	-3.1%	-2.2%	-2.3%	-2.2%	-2.4%	-2.2%
ING Groep NV	FN	2.6%	1.0	NA	-16%	55%	-2.1%	0.0%	2.6%	3.1%	0.2%	-2.6%	1.6%	1.6%	1.2%	-0.1%
BNP Paribas SA Class A	FN	3.0%	0.7	NA	4%	48%	2.1%	0.0%	-2.6%	3.1%	2.9%	4.0%	1.2%	-3.0%	1.5%	-3.0%
Unibail-Rodamco SE	FN	0.9%	1.3	88%	7%	-1%	-2.1%	0.0%	-2.6%	-3.1%	-3.2%	-0.9%	-3.4%	-0.9%	-3.3%	-0.9%
Fresenius SE & Co. KGaA	HC	1.3%	3.2	32%	8%	17%	0.0%	-3.2%	0.0%	0.0%	-2.0%	-1.3%	-1.4%	-1.3%	-1.7%	-1.3%
Bayer AG	HC	3.9%	2.7	57%	10%	36%	0.0%	0.0%	0.0%	3.1%	1.6%	4.6%	1.4%	3.5%	1.4%	5.0%
Sanofi	HC	4.0%	1.7	64%	2%	19%	2.1%	3.2%	2.6%	0.0%	3.6%	5.0%	3.5%	5.0%	3.6%	5.0%
Essilor International SA	HC	1.1%	3.5	59%	10%	-2%	-2.1%	0.0%	-2.6%	-3.1%	-3.2%	-1.1%	-3.4%	-1.1%	-3.3%	-1.1%
Airbus SE	ID	1.7%	13.3	8%	5%	50%	-2.1%	-3.2%	-2.6%	3.1%	-2.1%	-1.7%	-2.1%	-1.7%	-2.2%	-1.7%
Safran S.A.	ID	1.2%	4.4	23%	8%	37%	-2.1%	-3.2%	-2.6%	3.1%	-2.1%	-1.2%	-2.1%	-1.2%	-2.2%	-1.2%
Deutsche Post AG	ID	1.4%	3.4	44%	1%	30%	-2.1%	3.2%	2.6%	-3.1%	-0.9%	-1.4%	0.3%	-1.4%	0.1%	-1.4%
VINCI SA	ID	1.8%	2.2	13%	9%	22%	0.0%	-3.2%	-2.6%	-3.1%	-3.7%	-1.8%	-4.0%	-1.8%	-4.0%	-1.8%
Schneider Electric SE	ID	1.6%	1.8	38%	-2%	28%	2.1%	3.2%	2.6%	-3.1%	2.1%	3.1%	2.1%	5.0%	2.2%	5.0%
Siemens AG	ID	4.2%	2.5	30%	4%	36%	0.0%	0.0%	0.0%	0.0%	0.0%	-4.2%	0.0%	-4.2%	0.0%	-4.2%
Royal Philips NV	ID	1.2%	2.1	43%	5%	42%	2.1%	3.2%	0.0%	3.1%	5.1%	5.0%	3.7%	5.0%	4.1%	5.0%
Compagnie de Saint-Gobain SA	ID	1.0%	1.3	26%	-2%	36%	2.1%	0.0%	2.6%	0.0%	1.6%	-1.0%	2.0%	5.0%	1.9%	5.0%
SAP SE	IT	3.8%	3.8	72%	7%	27%	0.0%	3.2%	2.6%	3.1%	3.7%	5.0%	4.0%	5.0%	4.0%	5.0%
Nokia Oyj	IT	1.3%	1.3	36%	114%	12%	2.1%	-3.2%	-2.6%	-3.1%	-2.2%	-1.3%	-3.0%	-1.3%	-2.9%	-1.3%
ASML Holding NV	IT	2.1%	4.7	43%	29%	26%	-2.1%	0.0%	0.0%	0.0%	-1.5%	-2.1%	-0.9%	-2.1%	-1.0%	-2.1%
Air Liquide SA	MA	1.8%	2.5	36%	53%	23%	0.0%	3.2%	-2.6%	0.0%	1.9%	2.3%	0.3%	-1.8%	0.8%	-1.8%
CRH Plc	MA	1.1%	2.0	33%	-1%	22%	2.1%	0.0%	2.6%	-3.1%	0.1%	-1.1%	0.7%	-1.1%	0.5%	-1.1%
BASF SE	MA	3.2%	2.6	31%	7%	23%	-2.1%	-3.2%	0.0%	3.1%	-1.9%	-3.2%	-1.0%	-3.2%	-1.3%	-3.2%
Orange SA	TS	1.2%	1.5	58%	4%	4%	2.1%	3.2%	-2.6%	-3.1%	1.8%	4.2%	-0.2%	-1.2%	0.4%	-1.2%
Telefonica SA	TS	1.8%	2.4	16%	3%	12%	0.0%	-3.2%	0.0%	0.0%	-2.0%	-1.8%	-1.4%	-1.8%	-1.7%	-1.8%
Deutsche Telekom AG	TS	2.1%	2.6	35%	3%	13%	-2.1%	0.0%	2.6%	3.1%	0.2%	-2.1%	1.6%	5.0%	1.2%	3.6%
E.ON SE	UT	0.8%	1.1	7%	-43%	10%	0.0%	-3.2%	2.6%	0.0%	-1.9%	-0.8%	-0.3%	-0.8%	-0.8%	-0.8%
Enel SpA	UT	1.6%	1.2	49%	-3%	24%	-2.1%	3.2%	0.0%	3.1%	2.1%	3.0%	1.9%	5.0%	2.0%	5.0%
ENGIE SA	UT	1.0%	0.7	23%	-1%	-1%	2.1%	0.0%	0.0%	-3.1%	-0.1%	-1.0%	-0.5%	-1.0%	-0.4%	-1.0%
Iberdrola SA	UT	1.7%	1.0	19%	2%	17%	0.0%	0.0%	-2.6%	0.0%	-0.1%	-1.7%	-1.1%	-1.7%	-0.9%	-1.7%

Table 14: Application of the approach starting from the factors all the way to the active portfolio weights for two fully invested portfolios: unconstrained and long-only constrained. Three cases of factor allocation considered as described in the text: Maximum Diversification (MD), Equal Risk Budgeting (ERB) and Equal Risk Contribution (ERC). The Stoxx 50 index constituents were considered on July 21st 2017. The stock market capitalization weights are indicated. Four factors used: HML, RMW, CMA and MOM. Data source: FactSet and Worldscope.

In Table 14, we show the company names of the stocks in the Stoxx 50 index organized by sector. CD for Consumer Discretionary, CS for Consumer Staples, EN for Energy, FN for Financials, HC for Healthcare, ID for Industrials, IT for Information Technology, MA for Materials, TS for Telecommunication Services and UT for Utilities. We did not separate Real Estate from Financials as is now the case in the recent GICS definitions. We also include the price-to-book used to construct the

HML factor, the gross margin used for RMW, the asset growth used for CMA and the 11-month stock return calculated one month prior to the portfolio formation for MOM. In the columns 'Long-short factor portfolios', we show how the long-short factor portfolios **P** were constructed from the factor data. We enforced sector neutrality. Thus, in each sector, we allocated a positive weight to about one-third of the stocks that rank the highest for each respective factor and we allocated a negative weight to about one-third of the stocks ranked lowest by the respective factor. Each long-short factor portfolio is cash neutral and stocks are equally weighted. RMW does not apply to Financials. For HML, the price-to-book of stocks is ranked from the cheapest as the lowest to the most expensive as the highest. Similar, for CMA, stocks with lower asset growth are preferred. For RMW and UMD, higher values of gross margin are preferred and higher performance of stocks is preferred, respectively.

The $\boldsymbol{\Sigma}$ is calculated from three years of history of monthly stock returns and used to find the leverage required to set the ex-ante volatility of each long-short portfolio to 3%. We also calculated the factor correlation matrix $\boldsymbol{\rho}_f$ using historical data and the long-short factor portfolios **P.**

Using these long-short portfolios we then followed the approach described in Section 3 and calculated the factor risk budgets **RB** for the three cases. The information ratio of all factors was set to 0.5 and λ to 1. For MD we use equation (15) and for ERB equation (17). For ERC we numerically solved equation (19). Details on how to numerically solve for the factor risk budgets so that the contribution of each factor towards the total volatility of the multi-factor long-short portfolio aggregation can be found in Maillard et al. (2010) and in Leote de Carvalho et al. (2012).

Factors	MD			ERB			ERC		
	Weight	Exposure		Weight	Exposure		Weight	Exposure	
	unconstrained		Long only	unconstrained		Long only	unconstrained		Long only
HML	73%	53%	37%	44%	50%	29%	51%	49%	29%
RMW	62%	53%	58%	44%	42%	52%	52%	48%	51%
CMA	5%	53%	35%	44%	77%	47%	35%	71%	43%
UMD	50%	53%	44%	44%	57%	30%	45%	56%	32%

Table 15: Factor weights and exposures.

The factor risk budgets obtained can be found in Table 15. Here, we include a column with factor weights for the unconstrained portfolio and two for the univariate factor active exposures, the first for the unconstrained portfolio and the second for the long-only constrained portfolio.

Company name	GICS Sector	Factor exposure multivariate				Factor exposure univariate				Stock excess returns	
		HML	RMW	CMA	MOM	HML	RMW	CMA	MOM	MD	ERB
Adidas AG	CD	103%	90%	-192%	237%	-46%	106%	-85%	195%	3.6%	2.6%
LVMH Moet Hennessy Louis Vuitton SE	CD	42%	53%	205%	84%	120%	60%	246%	136%	5.8%	8.4%
Vivendi SA	CD	-36%	-109%	210%	23%	93%	-94%	196%	56%	1.3%	3.8%
Volkswagen AG Pref	CD	339%	-535%	403%	-231%	704%	-659%	514%	-279%	-0.4%	4.2%
Daimler AG	CD	103%	-323%	245%	37%	302%	-340%	303%	22%	0.9%	4.3%
Bayerische Motoren Werke AG	CD	258%	-303%	198%	66%	424%	-353%	340%	24%	3.3%	6.5%
Industria de Diseno Textil, S.A.	CD	38%	-43%	52%	11%	74%	-51%	73%	10%	0.9%	1.6%
L'Oreal SA	CS	-142%	-39%	161%	60%	-59%	9%	104%	108%	0.6%	2.4%
Anheuser-Busch InBev SA/NV	CS	-76%	-31%	41%	-111%	-34%	-32%	-23%	-98%	-2.7%	-2.8%
Royal Ahold Delhaize N.V.	CS	138%	-117%	-150%	-28%	95%	-157%	-89%	-102%	-2.4%	-3.8%
Danone SA	CS	-16%	-47%	60%	61%	18%	-32%	66%	68%	0.9%	1.8%
Unilever NV Cert. of shs	CS	-255%	-201%	156%	76%	-136%	-121%	44%	108%	-3.4%	-1.6%
Total SA	EN	-17%	-146%	196%	84%	108%	-125%	206%	105%	1.8%	4.4%
Eni S.p.A.	EN	119%	-93%	209%	46%	240%	-112%	278%	62%	4.2%	7.0%
Intesa Sanpaolo S.p.A.	FN	586%	-32%	-88%	306%	512%	-123%	277%	205%	11.6%	13.1%
Allianz SE	FN	40%	-151%	167%	35%	157%	-153%	194%	41%	1.4%	3.6%
Munich Reinsurance Company	FN	21%	-38%	98%	-22%	82%	-47%	103%	-9%	0.9%	1.9%
Banco Bilbao Vizcaya Argentaria, S.A.	FN	415%	-82%	218%	330%	502%	-124%	502%	313%	13.2%	17.9%
Banco Santander S.A.	FN	460%	-102%	320%	298%	608%	-159%	619%	295%	14.6%	20.4%
Deutsche Bank AG	FN	573%	-233%	268%	162%	745%	-345%	590%	108%	11.5%	16.5%
Societe Generale S.A. Class A	FN	673%	-98%	52%	433%	669%	-187%	489%	341%	15.9%	19.7%
AXA SA	FN	216%	-151%	99%	119%	288%	-182%	233%	86%	4.2%	6.4%
ING Groep NV	FN	221%	-88%	199%	224%	314%	-100%	361%	226%	8.3%	12.0%
BNP Paribas SA Class A	FN	589%	-88%	5%	415%	561%	-159%	396%	326%	13.8%	16.8%
Unibail-Rodamco SE	FN	-127%	-1%	113%	-112%	-56%	11%	23%	-70%	-1.9%	-1.4%
Fresenius SE & Co. KGaA	HC	-148%	-152%	74%	-66%	-64%	-126%	-16%	-58%	-4.4%	-4.0%
Bayer AG	HC	-76%	-182%	131%	87%	25%	-146%	112%	94%	-0.6%	1.3%
Sanofi	HC	33%	19%	208%	-28%	136%	7%	219%	20%	3.5%	5.7%
Essilor International SA	HC	-125%	-85%	58%	-45%	-69%	-62%	-16%	-31%	-3.0%	-2.7%
Airbus SE	ID	77%	-447%	-68%	237%	125%	-423%	22%	129%	-3.0%	-2.2%
Safran S.A.	ID	23%	-316%	-60%	181%	49%	-289%	-9%	106%	-2.6%	-2.1%
Deutsche Post AG	ID	-113%	-78%	333%	-45%	79%	-54%	265%	33%	1.5%	4.8%
VINCI SA	ID	71%	-71%	-33%	-9%	73%	-90%	0%	-38%	-0.6%	-0.8%
Schneider Electric SE	ID	145%	-22%	239%	-12%	272%	-58%	309%	22%	5.3%	8.1%
Siemens AG	ID	147%	-105%	114%	71%	221%	-128%	203%	59%	3.4%	5.3%
Royal Philips NV	ID	260%	-32%	22%	146%	260%	-70%	185%	112%	5.9%	7.3%
Compagnie de Saint-Gobain SA	ID	302%	-96%	98%	110%	361%	-150%	274%	77%	6.2%	8.4%
SAP SE	IT	20%	-20%	122%	37%	81%	-17%	140%	59%	2.4%	4.0%
Nokia Oyj	IT	294%	-216%	-270%	11%	211%	-290%	-123%	-129%	-2.7%	-5.0%
ASML Holding NV	IT	-145%	-208%	238%	59%	18%	-159%	177%	94%	-0.9%	2.0%
Air Liquide SA	MA	51%	0%	124%	83%	102%	4%	169%	106%	3.9%	5.7%
CRH Plc	MA	252%	-123%	128%	-66%	355%	-197%	238%	-90%	2.9%	4.6%
BASF SE	MA	-17%	-195%	299%	90%	170%	-171%	310%	126%	2.7%	6.5%
Orange SA	TS	88%	52%	-12%	-46%	75%	22%	22%	-50%	1.2%	1.0%
Telefonica SA	TS	108%	-164%	137%	165%	197%	-160%	228%	153%	3.7%	6.3%
Deutsche Telekom AG	TS	-83%	-57%	161%	-22%	15%	-39%	114%	15%	0.0%	1.6%
E.ON SE	UT	-154%	-336%	580%	10%	219%	-290%	502%	103%	1.5%	8.0%
Enel SpA	UT	33%	-23%	49%	107%	50%	-11%	91%	110%	2.5%	3.6%
ENGIE SA	UT	60%	-106%	185%	-43%	185%	-127%	204%	-27%	1.5%	3.5%
Iberdrola SA	UT	46%	-4%	25%	34%	55%	-9%	56%	33%	1.5%	2.0%

Table 16: Expected stock excess returns derived from the factor exposures.

The vector with the risk budget of each factor **RB** is then used in equation (9) along with the matrix **P** to calculate the unconstrained optimal stock active weights shown in Table 14. The constrained portfolios are calculated numerically from equation (8) from the variance-covariance matrix **Σ** and the stock expected returns **R** while enforcing that the sum of stock active weights must be zero and that no short or leveraged positions are allowed for any stock. For this, the expected returns are calculated first using (16), (18) and (20) for MD, ERB and ERC respectively. The stock expected returns for the MD and ERB cases are given in Table 16 along with the univariate and multivariate betas. It is simple to observe that they follow from equations (16) and (18), respectively.

The objective of Table 14 is to show the optimal constrained and unconstrained multi-factor portfolios. We include the weights of each stock in the market capitalization index and the active weights in each case of factor allocation considered. It is interesting to note that the cross-sectional correlation of active weights for the unconstrained and constrained portfolios in Table 14 is 80% for MD and 75% for both ERB and ERC. This demonstrates empirically that despite the strong long-only constraint, the optimizer still delivers active weights that remain relatively close to the unconstrained case. Finally, from table 15 we can see by how much the portfolio constraints pushed factor exposures away from the targeted risk budget, which gives a clear indication of the impact of constraints on the factor exposures. In all three cases, we found an under-exposure to HML, CMA and UMD and to over-exposure to RMW in the constrained portfolios.

5. CONCLUSIONS

In this paper, we discuss the question of multifactor portfolio construction and show that simplistic approaches often used by practitioners tend to be sub-optimal. In the first part of the paper, we focus on three approaches to generate stock expected returns from factor expected returns with the objective of using these as inputs in mean-variance optimization for the construction of realistic portfolios with targeted multi-factor exposures. We demonstrate that the two most simplistic approaches generate portfolios with a significant percentage of the tracking error allocated to unintended exposures to factors that are orthogonal to those targeted. Indeed, we show that stock expected returns must be derived in a robust fashion from factor expected returns if we wish to use them in mean-variance optimization and make sure that portfolios have only exposures to the targeted factors. We discuss at length the framework that we recommend for this purpose. In the second part of the paper, we provide a detailed example to show how to apply the framework and we demonstrate its efficiency for the construction of realistic constrained portfolios, e.g. long-only. We show that even when applying constraints, the portfolios retain much of the targeted factor exposures while minimize the impact of constraints.

Building portfolios with multiple factor exposures for factor investing is certainly not easy, in particular when realistic constraints faced by practitioners and investors apply. We believe that our paper makes an important contribution towards highlighting the problems in factor investing and proposing solutions, in particular for benchmarked constrained factor investing in equities.

6. APPENDIX

Let's consider the risk model (6). This was written in terms of the expected stocks returns and expected factor returns at a given point in time. But the estimation of the stock exposures to factors $\boldsymbol{\beta}$ typically relies on the historical time series of stock returns. If $\mathbf{r}(T * N)$ is a matrix with the time series of returns over T periods for the N stocks and $\mathbf{f}(T * K)$ is a matrix with the time series of returns over T periods for the K factors, then the multifactor exposures are determined from:

$$\mathbf{r} = \mathbf{f}\,\boldsymbol{\beta}_m + \boldsymbol{\varepsilon} \tag{A1}$$

Using the matrix $\mathbf{P}(N * K)$, the zero sum long-short weights for N stocks in each of the K factors

$$\mathbf{r} = \mathbf{rP}\,\boldsymbol{\beta}_m + \boldsymbol{\varepsilon} \tag{A2}$$

We can derive equation (12) by applying an ordinary least square regression to (A2):

$$\boldsymbol{\beta}_m = \mathbf{r}^T\mathbf{rP}\,(\mathbf{P}^T\mathbf{r}^T\mathbf{rP})^{-1}$$

$$\propto \boldsymbol{\Sigma}\,\mathbf{P}\,\boldsymbol{\Theta}^{-1}$$

$$\propto \boldsymbol{\Sigma}\,\mathbf{P}\,\boldsymbol{\rho}_f^{-1} \tag{A3}$$

Where $\boldsymbol{\Sigma}$ is the stock variance-covariance matrix estimated from the historical times series of returns, $\boldsymbol{\Theta}$ is the factor variance-covariance matrix estimated from the historical times series of returns of the portfolios $\mathbf{P}$, and $\boldsymbol{\rho}_f$ is the factor correlation matrix. In the last step, it was assumed that all long-short factor portfolios in $\mathbf{P}$ have the same volatility.

7. DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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